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TEMPERATURE SHOCK WAVES IN A MOVING MEDIUM WITH ALLOWANCE FOR THE RELAXATION OF THE HEAT FLUX

P. P. Volosevich, E. I. Levanov, and E. V. Severina

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The solution of the gasdynamic equation with allowance for the heat transfer in the relaxation of the heat flux is analyzed. The relations expressing the laws of conservation on the front of strong discontinuity of the quantities sought, including the discontinuity of the temperature and the heat-flux density, are discussed. The possibility of existence of two shock waves with fixed initial data is shown using the self-similar solution of the problem on gas motion ahead of the piston. The occurrence of two strong discontinuities is due to the presence of different velocities of propagation of gasdynamic and thermal disturbances — the velocity of sound and the finite rate of heat transfer at a nonzero time of relaxation of the heat flux.

To investigate the processes of heat transfer one frequently uses the representation of the heat flux, which it is in proportion to the temperature gradient (Fourier law). However, situations where the scales of space-time inhomogeneities are comparable to the mean path and the characteristic time between electron collisions are not uncommon in many physical problems. Description of the heat flux using the Fourier law becomes inapplicable in these cases. When the temperature gradients are large, the Fourier model may exceed the limiting value of the heat flux, which corresponds to coordinated electronic motion in one direction. An attempt at allowing for the constraint imposed on the heat flux by physically unjustified interpolation formulas leads to substantially distorted results [1].

In the present work, the expression of the heat flux is considered with allowance for the term dependent on the time derivative of the heat flux [2-4]. In Lagrangian mass variables m and t, we may write an expression allowing for the heat-flux relaxation for the case of plane symmetry in the form

$$W = -K\frac{\partial T}{\partial m} - \tau\frac{\partial W}{\partial t}\,. \tag{1}$$

Equation (1), particularly in the case of a stationary medium and for constant values of the parameters K and τ, has been the focus of an extensive amount of literature (see [3–7] and references therein)]. Together with the corresponding energy equation it represents a hyperbolic system and, consequently, strong discontinuities of the functions $T = T(m,\ t)$ and $W = W(m,\ t)$, i.e., shock thermal waves may exit.

Heat transfer with allowance for the motion of a medium is described by a system of gasdynamic equations, which, in Lagrangian mass variables m and t and in the plane-symmetry approximation, has the form

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \frac{\partial v}{\partial m}\,, \tag{2}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial m}\,, \tag{3}$$

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{1}{2}v^2\right) = -\frac{\partial}{\partial m}(pv + W)\,. \tag{4}$$

Institute of Mechanics and Mathematics, Russian Academy of Sciences, 4a Miusskaya Sq., Moscow, 125047, Russia; email: levanov@imamod.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 79, No. 4, pp. 57–68, July–August, 2006. Original article submitted November 11, 2005.

The heat flux W satisfies Eq. (1) which may be written (at $\tau \neq 0$) in the form

$$\frac{\partial W}{\partial t} = -\frac{K}{\tau}\frac{\partial T}{\partial m} - \frac{W}{\tau}. \quad (5)$$

In the general case the thermophysical quantities K and τ, the pressure p, and the specific internal energy ε are arbitrary functions of the density ρ and the temperature T

$$K = K(\rho, T), \quad \tau = \tau(\rho, T), \quad p = p(\rho, T), \quad \varepsilon = \varepsilon(\rho, T). \quad (6)$$

Laws of Conservation on the Front of a Thermal Shock Wave. The relations expressing the laws of conservation on the strong-discontinuity front analogously to [8] may be obtained by formal integration of the system of Eqs. (2)–(5) in the small vicinity of the discontinuity surface.

Let $m_j = m_j(t)$ be the discontinuity surface in the plane (m, t) passing through the points with coordinates $(m = m_1,\ t = t_1)$ and $(m = m_2,\ t = t_2)$, where $m_2 = m_1 + \Delta m$ and $t_2 = t_1 + \Delta t$; Δm and Δt are the small quantities. Integrating Eqs. (2)–(5) going from m_1 to m_2 and from t_1 to t_2, we obtain

$$\int_{m_1}^{m_2}\left[\left(\frac{1}{\rho}\right)_{t_2} - \left(\frac{1}{\rho}\right)_{t_1}\right] dm = -\int_{t_1}^{t_2}\left[(v)_{m_2} - (v)_{m_1}\right] dt, \quad (7)$$

$$\int_{m_1}^{m_2}\left[(v)_{t_2} - (v)_{t_1}\right] dm = -\int_{t_1}^{t_2}\left[(p)_{m_2} - (p)_{m_1}\right] dt, \quad (8)$$

$$\int_{m_1}^{m_2}\left[\left(\varepsilon + \frac{v^2}{2}\right)_{t_2} - \left(\varepsilon + \frac{v^2}{2}\right)_{t_1}\right] dm = -\int_{t_1}^{t_2}\left[(pv + W)_{m_2} - (pv + W)_{m_1}\right] dt, \quad (9)$$

$$\int_{m_1}^{m_2}\left[(W)_{t_2} - (W)_{t_1}\right] dm = -\int_{t_1}^{t_2}\int_{m_1}^{m_2}\frac{K}{\tau}\frac{\partial T}{\partial m}\, dm dt - \int_{t_1}^{t_2}\int_{m_1}^{m_2}\frac{W}{\tau}\, dm dt. \quad (10)$$

To represent Eq. (10) in an integral form analogous to Eqs. (7)–(9) in structure we introduce the auxiliary function $V = V(m, t)$ satisfying (at $\tau \neq 0$) the equation

$$\frac{\partial V}{\partial m} = \frac{K(\rho, T)}{\tau(\rho, T)}\frac{\partial T}{\partial m}. \quad (11)$$

With account for (11), Eq. (10) will take the form

$$\int_{m_1}^{m_2}\left[(W)_{t_2} - (W)_{t_1}\right] dm = -\int_{t_1}^{t_2}\left[(V)_{m_2} - (V)_{m_1}\right] dt - \int_{t_1}^{t_2}\int_{m_1}^{m_2}\frac{W}{\tau}\, dm dt. \quad (12)$$

The function V is determined from Eq. (11) in solving specific problems with prescribed expressions of the parameters K and τ. In particular, it is assumed in [4, 6, 9] that the ratio $K/\tau = \Phi$ is either constant or dependent just on temperature. In this case we obtain $\partial V/dT = K/\tau = \Phi(T)$ from (11); the function V may be determined, analogously to [4, 6, 9], by the relation

$$V=\int_0^T \Phi(T)\,dT\,. \tag{13}$$

If the parameters K and τ are power functions of T and ρ

$$K=K_0T^{a_0}\rho^{b_0+1}\,,\ \ \tau=\tau_0T^{a_1}\rho^{b_1}\,, \tag{14}$$

then, integrating (13) with the conditions

$$b_0+1-b_1=0\,,\ \ a_0-a_1+1\neq 0\,, \tag{15}$$

we obtain

$$V=\frac{K_0}{\tau_0\,(a_0-a_1+1)}\,T^{a_0-a_1+1}\,. \tag{16}$$

We note that conditions (15) are satisfied, for example, by the ratio of the mass radiative thermal conductivity $K = K_0T^{13/2}\rho^{-1}$ to the heat-flux-relaxation time $\tau = \tau_0T^{3/2}\rho^{-1}$ determined by electronic collisions.

Following [8], we use the mean-value theorem for each individual term of Eqs. (7)–(9) and (12). Let t_*, t_{**}, $\tilde{t}$, m_*, m_{**}, and $\tilde{m}$ be the mean values of the arguments m and t, lying between the values m_1 and m_2 and t_1 and t_2 respectively. Then we obtain

$$\left[\left(\frac{1}{\rho}\right)_{t_2m_{**}}-\left(\frac{1}{\rho}\right)_{t_1m_*}\right]\Delta m=\left[(v)_{t_{**}m_2}-(v)_{t_*m_1}\right]\Delta t\,, \tag{17}$$

$$\left[(v)_{t_2m_{**}}-(v)_{t_1m_*}\right]\Delta m=-\left[(p)_{t_{**}m_2}-(p)_{t_*m_1}\right]\Delta t\,, \tag{18}$$

$$\left[\left(\varepsilon+\frac{v^2}{2}\right)_{t_2m_{**}}-\left(\varepsilon+\frac{v^2}{2}\right)_{t_1m_*}\right]\Delta m=-\left[(W+pv)_{t_{**}m_2}-(W+pv)_{t_*m_1}\right]\Delta t\,, \tag{19}$$

$$\left[(W)_{t_2m_{**}}-(W)_{t_1m_*}\right]\Delta m=-\left[(V)_{t_{**}m_2}-(V)_{t_*m_1}\right]\Delta t-\left(\frac{W}{\tau}\right)_{\tilde{t}\tilde{m}}\Delta m\Delta t\,. \tag{20}$$

Passing to the limit at $\Delta m\to 0$ and $\Delta t\to 0$ and determining the Lagrangian mass velocity of propagation of the discontinuity front from the formula

$$D=\lim_{\Delta t\to 0}\frac{\Delta m}{\Delta t}=\frac{dm_j}{dt}\,, \tag{21}$$

we obtain from (17)–(20) that

$$\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)D=v_1-v_2\,,\ \ (v_2-v_1)\,D=p_2-p_1\,,\ \ \left(\varepsilon_2+\frac{v_2^2}{2}-\varepsilon_1\frac{v_1^2}{2}\right)D=p_2v_2-p_1v_1+W_2-W_1\,,$$
$$(W_2-W_1)\,D=V_2-V_1\,. \tag{22}$$

Setting

$$\theta = \frac{\rho_1}{\rho}, \tag{23}$$

we may write formulas (22) in the form

$$\rho_2 = \frac{\rho_1}{\theta_2}, \tag{24}$$

$$v_2 = v_1 + (1 - \theta_2)\,\frac{D}{\rho_1}, \tag{25}$$

$$p_2 = p_1 + (1 - \theta_2)\,\frac{D^2}{\rho_1}, \tag{26}$$

$$W_2 = W_1 + \left[\varepsilon_2 - \varepsilon_1 - \frac{p_1}{\rho_1}(1 - \theta_2)\right] D - \frac{1}{2}(1 - \theta_2)^2\,\frac{D^3}{\rho_1^2}, \tag{27}$$

$$V_2 = V_1 + (W_2 - W_1)\,D\,. \tag{28}$$

Relations (25)–(28) express the laws of conservation in passage of the quantities sought through the front of the shock thermal wave. For the known expression of the function $V = V(m, t)$ and the equations of state of the medium $p = p(\rho, T)$ and $\varepsilon = \varepsilon(\rho, T)$ these relations determine the values of thermal and gasdynamic quantities behind the discontinuity front in terms of the velocity D and the corresponding values ahead of the front.

The gas equations

$$p = \rho R T\,, \quad \varepsilon = \frac{RT}{\gamma - 1} \tag{29}$$

will be assumed to hold true. An expression for the heat flux $W = W_2$ with account for (29) may be written in the form

$$W_2 = W_1 + \frac{1}{2}\,\frac{\gamma + 1}{\gamma - 1}(1 - \theta_2)\,D\left[\left(\theta_2 - \frac{\gamma - 1}{\gamma + 1}\right)\frac{D^2}{\rho_1^2} - \frac{2\gamma}{\gamma + 1}\,\frac{p_1}{\rho_1}\right]. \tag{30}$$

Also, we write the values of the function of temperature $T = T_2$ behind the discontinuity front

$$T_2 = \frac{p_2}{R\rho_2} = \theta_2\left[T_1 + (1 - \theta_2)\,\frac{D^2}{R\rho_1^2}\right]. \tag{31}$$

In the case (16) expression (28) yields

$$\frac{K_0}{\tau_0\,(a_0 - a_1 + 1)}\left(T_2^{a_0 - a_1 + 1} - T_1^{a_0 - a_1 + 1}\right) = \frac{1}{2}\,\frac{\gamma + 1}{\gamma - 1}(1 - \theta_2)\,D^2\left[\left(\theta_2 - \frac{\gamma - 1}{\gamma + 1}\right)\frac{D^2}{\rho_1^2} - \frac{2\gamma}{\gamma + 1}\,\frac{p_1}{\rho_1}\right]. \tag{32}$$

When $v_1 = p_1 = T_1 = W_1 = 0$ and $\rho = \rho_1$ Eq. (32) has the form [4]

$$\theta_2 - \frac{\gamma-1}{\gamma+1} = A_0 \theta_2^{a_0 - a_1 + 1} (1-\theta_2)^{a_0 - a_1} , \tag{33}$$

where

$$A_0 = \frac{2(\gamma-1) K_0}{(\gamma+1)(a_0 - a_1 + 1) \tau_0} D^{2(a_0 - a_1 - 1)} R^{-a_0 + a_1 - 1} \rho_1^{-2(a_0 - a_1)}$$

is the constant.

Two Strong Discontinuities. The velocity of sound and the velocity (finite at $\tau \neq 0$) of propagation of thermal disturbances are the characteristic velocities in gas dynamics allowing for the relaxation of the heat flux. The mass velocity of transfer of heat is determined by the relation $C_T = \sqrt{(\gamma-1)K/R\tau}$, and the velocity of sound is determined by the formula $C_\gamma = \rho\sqrt{\gamma RT}$ in the case where the gas equations (29) hold true. The presence of two "velocities of sound" may generate two strong discontinuities of gasdynamic and thermal quantities [3, 4]. The possibility of two temperature shock waves existing with fixed initial conditions will be shown using an analysis of the self-similar solution of the piston problem.

Formulation of the Problem. Let the thermal regime and the piston velocity vary with time in the plane $m = 0$ by the power law

$$T(0, t) = T_0 t^{n_0} , \quad v(0, t) = v_0 t^{n_1} , \quad T_0 > 0 , \quad v_0 > 0 . \tag{34}$$

At $t = 0$, the conditions

$$T(m, 0) = v(m, 0) = W(m, 0) = 0 , \quad \rho(m, 0) = \rho_0 = \text{const} \tag{35}$$

are observed. Let us assume next that

$$K = K_0 T^{a_0} , \quad \tau = \tau_0 T^{a_1} . \tag{36}$$

The solution of problem (1)–(4), (29), and (34)–(36) is self-similar with the conditions

$$a_0 = 1 + \frac{1}{n_0} , \quad a_1 = a_0 - 1 , \quad n_0 \neq 0 , \quad n_1 = \frac{1}{2} n_0 . \tag{37}$$

Let a_1 be more than 0 and, consequently, $a_0 > 1$ and $n_0 > 0$. By replacement of variables of the form

$$\begin{gathered} s = \frac{m}{\rho_0 (RT_0)^{1/2} t^n} , \quad \alpha(s) = \frac{v(m, t)}{(RT_0)^{1/2} t^{\frac{1}{2}n_0}} , \quad \delta(s) = \frac{\rho(m, t)}{\rho_0} , \\ f(s) = \frac{T(m, t)}{T_0 t^{n_0}} , \quad \omega(s) = \frac{W(m, t)}{\rho_0 (RT_0)^{3/2} t^{\frac{3}{2}n_0}} , \quad \beta(s) = \frac{p(m, t)}{\rho_0 R T_0 t^{n_0}} , \end{gathered} \tag{38}$$

where $n = \frac{1}{2} n_0 + 1$, we reduce Eqs. (1)–(4) with account for (36) and (37) to the following system of ordinary differential equations of first order in independent variable $s \geq 0$ (the derivative with respect to s is primed):

$$\alpha' = \frac{ns}{\delta^2} \delta' , \tag{39}$$

$$\frac{1}{2} n_0 \alpha - ns\alpha' = -\beta' , \tag{40}$$

$$\frac{1}{\gamma-1}(n_0 f - nsf') = -\delta f\alpha' - \omega', \tag{41}$$

$$\omega = -\tilde{K}f' - \tilde{\tau}\left(\frac{3}{2}n_0\omega - ns\omega'\right), \tag{42}$$

where

$$\beta = f\delta; \quad \hat{K} = \hat{K}_0 f^{a_0}; \quad \tilde{\tau} = \hat{\tau}_0 f^{a_0-1}; \tag{43}$$

$$\hat{K}_0 = K_0 R^{-2} T_0^{a_0-1} \rho_0^{-2}; \quad \hat{\tau}_0 = \tau_0 T_0^{a_0-1} \tag{44}$$

are the dimensionless constants.

Boundary conditions on the piston (34) in the variables of (38) take the form

$$f(0) = 1, \quad \alpha(0) = \alpha_0 = \frac{v_0}{(RT_0)^{1/2}}. \tag{45}$$

Conditions (35) must be observed at the right end of the s axis. Let the variable s vary within $0 \le s \le s_0$, where $0 < s_0 < \infty$. Then for $s = s_0$ we obtain

$$f(s_0) = \alpha(s_0) = \omega(s_0) = 0, \quad \delta(s_0) = 1. \tag{46}$$

Qualitative Analysis of the Solution. 1. The solution of the problem in question is known [10–12] to describe a temperature wave of finite velocity for $\tau_0 = 0$ and $a_0 > 0$. The solution is continuous in the vicinity of $s = s_0$, $0 < s_0 < \infty$. The functions sought satisfy conditions (46) and are determined accurate to the principal terms in the vicinity of $s = s_0$ by the formulas

$$f = \left(\frac{s_0 n a_0}{(\gamma-1)\hat{K}_0}\right)^{1/a_0} (s_0 - s)^{1/a_0}, \quad \omega = \frac{ns_0}{\gamma-1} f, \quad \alpha = \frac{1}{ns_0} f, \quad \delta = 1 + \frac{1}{(ns_0)^2} f. \tag{47}$$

We have an isothermal discontinuity (discontinuity of the gasdynamic quantities and the heat flux at continuous temperature) at a certain point $s = s_1$, $0 < s_1 < s_0$ deep in the thermal-wave front. There are different types of solution characterized by either the advance of the heating-wave front or its lag behind the discontinuity front: the so-called TV-I and TV-II regimes [10–12].

Let us assume that there can also be a continuous solution in the vicinity of s_0 in the case $\tau \neq 0$. We set $s = s_0 - x$, where $x > 0$ is the small quantity. Preserving the principal terms, from Eqs. (39)–(41) we obtain expressions analogous to (47) for the functions ω, α, and δ. Substituting the function $\omega = \dfrac{ns_0}{\gamma-1} f$ into Eq. (42), we find as a first approximation

$$\frac{ns_0}{\gamma-1} f = \hat{K}_0 f^{a_0} \frac{df}{dx} - \hat{\tau}_0 n^2 s_0^2 \frac{1}{\gamma-1} f^{a_0-1} \frac{df}{dx}. \tag{48}$$

When $a_0 > 1$ the principal term on the right-hand side of (48) is the second term. The approximate solution of Eq. (48) will have the form

$$x = s_0 - s = C - \frac{\hat{\tau}_0 n s_0}{a_0 - 1} f^{a_0-1}, \tag{49}$$

where C is the integration constant. The condition $f(s_0) = 0$ is observed for $C = 0$. However, we must have $x \geq 0$ ($s \leq s_0$) in the small vicinity of $s = s_0$. Consequently, formula (49) has no physical meaning, i.e., the solution in the vicinity of $s = s_0$ cannot be continuous at least for $a_0 > 1$ ($a_1 > 0$). It only remains for us to assume that we have a strong discontinuity of the functions sought at a point characterized by the coordinate $s = s_0$, $0 < s_0 < \infty$. Condition (46) is observed ahead of the discontinuity front in the variables of (38). The front velocity is determined by the value $d(s_0) = ns_0$. Behind the discontinuity front, with account for self-similarity conditions (37) we obtain the relations

$$\theta(s_0) = \theta_0 = \frac{1}{\delta_2}, \tag{50}$$

$$\alpha(s_0) = \alpha_2 = (1 - \theta_0)\, ns_0, \tag{51}$$

$$\beta(s_0) = \beta_2 = (1 - \theta_0)\, n^2 s_0^2, \tag{52}$$

$$f(s_0) = f_2 = \theta_0 (1 - \theta_0)\, n^2 s_0^2, \tag{53}$$

$$\omega(s_0) = \omega_2 = \frac{1}{2}\left(\frac{\gamma + 1}{\gamma - 1}\theta_0 - 1\right)(1 - \theta_0)\, n^3 s_0^3 = \frac{1}{2}\frac{\hat{K}_0}{\hat{\tau}_0}\frac{f_2^2}{ns_0}. \tag{54}$$

Equation (33) in the variables of (38) by virtue of (37) will take the form

$$\theta_0 - \frac{\gamma - 1}{\gamma + 1} = \theta_0^2 (1 - \theta_0)\frac{\gamma - 1}{\gamma + 1}\frac{\hat{K}_0}{\hat{\tau}_0}. \tag{55}$$

2. To determine a possible structure of solution in the region between the piston ($s = 0$) and the discontinuity front ($s = s_0$) we solve the system of Eqs. (39)–(42) for the derivatives. The above-mentioned "velocities of sound" squared in the variables of (38) may be written in the form

$$\tilde{C}_T^2 = \frac{(\gamma - 1)\tilde{K}}{\tilde{\tau}} = (\gamma - 1) f \frac{\hat{K}_0}{\hat{\tau}_0}, \quad \tilde{C}_i^2 = \delta^2 f, \quad \tilde{C}_\gamma^2 = \gamma \tilde{C}_i^2. \tag{56}$$

In what follows we drop the ~ sign above the corresponding dimensionless values of the "velocities of sound" for the sake of simplification. We set

$$\varphi = \delta^2 \left(\frac{1}{2} n_0 \alpha (n^2 s^2 - C_T^2) + \frac{\delta(\gamma - 1)\,\omega}{f^{a_0 - 1}\tau_0} + \delta n_0 \left(nsf + \frac{3}{2}\omega(\gamma - 1)\right)\right) \tag{57}$$

and

$$\Delta = C_T^2 C_i^2 + n^2 s^2 (n^2 s^2 - (C_T^2 + \gamma C_i^2)). \tag{58}$$

Then Eqs. (39)–(43) will take the form

$$\delta' = \frac{\varphi}{\Delta}, \tag{59}$$

$$\alpha' = \frac{ns}{\delta^2}\delta', \tag{60}$$

$$f' = \frac{1}{\delta^3}\left[(n^2 s^2 - C_i^2)\,\delta' - \frac{1}{2} n_0 \delta^2 \alpha\right], \tag{61}$$

$$\omega' = \frac{1}{\gamma - 1}\left(\frac{ns}{\delta^3}\left((n^2 s^2 - \gamma C_i^2)\,\delta' - \frac{1}{2} n_0 \alpha \delta^2\right) - n_0 f\right). \tag{62}$$

After transformations, we may represent relation (58) in the form

$$\Delta = n^2 s^2\,[n^2 s^2 - (C_T^2 + \gamma C_i^2)] + C_i^2 C_T^2 = (n^2 s^2 - \lambda_+^2)\,(n^2 s^2 - \lambda_-^2)\,, \tag{63}$$

where the variables λ_+^2 and λ_-^2 are determined by the formulas

$$\lambda_\pm^2 = \frac{C_T^2 + \gamma C_i^2 \pm \sqrt{(C_T^2 - \gamma C_i^2)^2 + 4\,(\gamma - 1)\,C_T^2 C_i^2}}{2}\,. \tag{64}$$

We compare the signs of the relations $\Delta(0)$ and $\Delta(s_0)$. From (63), we obtain that on the piston $s = 0$ we have

$$\Delta\,(0) = \lambda_+^2 \lambda_-^2 > 0\,. \tag{65}$$

At the point $s = s_0$ characterizing the strong-discontinuity front, we have

$$\Delta\,(s_0) = (n^2 s_0^2 - \lambda_+^2\,(s_0))\,(n^2 s_0^2 - \lambda_-^2\,(s_0))\,. \tag{66}$$

Since λ_+ is more than λ_-, it follows from (66) that we have the inequality $\Delta(s_0) > 0$ for $ns_0 > \lambda_+(s_0)$ or for $ns_0 < \lambda_-(s_0)$. In the case

$$\lambda_-\,(s_0) < ns_0 < \lambda_+\,(s_0) \tag{67}$$

we obtain

$$\Delta\,(s_0) < 0\,. \tag{68}$$

The presence of different signs of the parameter Δ at the points $s = 0$ and $s = s_0$ with condition (67) means that the value $s = s_k$, $0 < s_k < \infty$ for which the condition $\Delta(s_k) = 0$ is observed must exist. An analysis performed analogously to [10, 11] shows that there can be no continuous passage through the point $s = s_k$, as we move in s from the value $s = s_0$ toward $s = 0$ or in the opposite direction: the continuous solution turns out to be nonunique. Indeed, let us assume that $\Delta(s_k) = 0$, and the parameter $\varphi = \varphi(s_k) = \varphi_k$ appearing in the numerator of Eq. (59) and accordingly Eqs. (60)–(62) is nonzero. Then the derivatives of all the quantities sought are equal to infinity at the point $s = s_k$.

We consider the system of Eqs. (57)–(62) in the vicinity of $s = s_k$. Let $\tilde{s} = s - s_k$ and $\tilde{y} = y - y_k$ be the small quantities where any of the quantities sought satisfying the system of Eqs. (57)–(62) is meant by y. Preserving only the principal terms, from (59) and in the vicinity of $\tilde{s} = 0$ we obtain

$$\frac{d\tilde{\delta}}{d\tilde{s}} = \frac{\varphi_k}{2n^4 s_k^3 \tilde{s} + A_k \tilde{f} + B_k \tilde{\delta}}\,, \tag{69}$$

where $A_k = (y - 1)\frac{\hat{K}_0}{\hat{\tau}_0}\left(2f_k\delta_k^2 - n^2 s_k^2\right) + \gamma n^2 s_k^2 \delta_k^2$ and $B_2 = 2\delta_k f_k\left(\gamma + (\gamma - 1)\,\frac{\hat{K}_0}{\hat{\tau}_0} f_k\right)$. From (60)–(62), after the corresponding integration, we have

$$\tilde{\alpha} = \frac{ns_k}{\delta_k^2}\,\tilde{\delta}\,, \tag{70}$$

$$\tilde{f} = \frac{1}{\delta_k^3}\,(n^2 s_k^2 - \delta_k^2 f_k)\,\tilde{\delta} - \frac{n_0 \alpha_k \tilde{s}}{2\delta_k}\,, \tag{71}$$

$$\tilde{\omega} = \frac{1}{\gamma - 1}\left[\frac{ns_k}{\delta_k^3}\,(n^2 s_k^2 - \delta_k^2 f_k)\,\tilde{\delta} - n_0\left(\frac{\alpha_k n s_k}{2\delta_k} + f_k\right)\tilde{s}\right]. \tag{72}$$

Taking next (71) into account, we reduce Eq. (69) to the form

$$\frac{d\tilde{s}}{d\tilde{\delta}} = \frac{1}{\varphi_k}\,(\psi_k \tilde{s} + \chi_k \tilde{\delta})\,, \tag{73}$$

where

$$\varphi_k = 2n^4 s_k^3 - \frac{n_0 \alpha_k A_k}{2\delta_k}\,;\quad \chi_k = B_k + \frac{A_k}{\delta_k^3}\,(n^2 s_k^2 - \delta_k^2 f_k)\,.$$

The general solution of the linear equation (73) is determined by the formula

$$\tilde{s} = C \exp\,(L_k \tilde{\delta}) - \frac{\chi_k}{\psi_k L_k}\,(L_k \tilde{\delta} + 1)\,, \tag{74}$$

where $L_k = \psi_k/\varphi_k$ and C is the integration constant. The conditions $\tilde{s} = 0$ and $\tilde{\delta} = 0$ are observed for $C = \chi_k/(\psi_k L_k)$. In this case (74), accurate to the principal terms, takes the form

$$\tilde{s} = \frac{\chi_k L_k}{\psi_k}\,\delta^2\,. \tag{75}$$

It follows from (75) that the point $\tilde{s} = 0$, $\tilde{\delta} = 0$ ($s = s_k$, $\delta = \delta_k$) at which we have $\Delta(s_k) = 0$ and $\varphi(s_k) \neq 0$ is the point of rotation of the integral curve $\delta = \delta(s)$. What this means is that the integral curves describing the dimensionless density function cannot be continuously extended from the region $s > s_k$ ($\Delta_k < 0$) to the region $s < s_k$ ($\Delta_k > 0$), and conversely. It follows from (70)–(72) that we also cannot continuously extend the remaining functions through the point $s = s_k$ analogously. Passage from the region $\Delta > 0$ to the region $\Delta < 0$ may be carried out through discontinuity of all the quantities sought. It is noteworthy that, since the dimensionless "velocity of state" $d(s_k) = ns_k$ coincides with one "velocity of sound" (λ_{+k} or λ_{-k}) at the point $s = s_k$, the coordinate characterizing the front of "internal" strong discontinuity is different from the value $s = s_k$: $s_1 \neq s_k$.

The functions sought on the second-discontinuity front which propagates over the disturbed background, not over the "zero" one, are determined by relations (25), (26), (28), (30), and (31). In the variables of (38), at the point $s = s_1$ ahead of the discontinuity front, we will have

$$\alpha\,(s_1) = \alpha_1\,,\;\; f\,(s_1) = f_1\,,\;\; \omega\,(s_1) = \omega_1\,,\;\; \delta\,(s_1) = \delta_1\,,\;\; \beta\,(s_1) = \beta_1\,. \tag{76}$$

Behind the discontinuity front moving with a velocity $d(s_1) = ns_1$, setting

$$\theta\,(s_1) = \theta_1 = \frac{\delta_1}{\delta_2}\,, \tag{77}$$

we obtain

$$\alpha(s_1) = \alpha_2 = \alpha_1 + (1-\theta_1)\, n s_1 \,, \tag{78}$$

$$\beta(s_1) = \beta_2 = \beta_1 + (1-\theta_1)\,\frac{n^2 s_1^2}{\delta_1}\,, \tag{79}$$

$$f(s_1) = f_2 = f_1\theta_1 + \theta_1\,(1-\theta_1)\,\frac{n^2 s_1^2}{\delta_1^2}\,, \tag{80}$$

$$\omega(s_1) = \omega_2 = \omega_1 + (1-\theta_1)\,\frac{n^3 s_1^3}{2\delta_1^2}\left(\frac{\gamma+1}{\gamma-1}\,\theta_1 - \left(1 + \frac{2\gamma\delta_1^2 f_1}{(\gamma-1)\,n^2 s_1^2}\right)\right). \tag{81}$$

Equation (32) in the variables of (38) with account for self-similarity conditions (37) has the form

$$(1-\theta_1)\left[\theta_1 - \frac{\gamma-1}{\gamma+1}\left(1 + \frac{2\gamma\delta_1^2 f_1}{(\gamma-1)\,n^2 s_1^2}\right)\right] = \frac{\gamma-1}{\gamma+1}\,\frac{\hat{K}_0}{\hat{\tau}_0}\,\frac{\delta_4^2}{n^4 s_1^4}\left[\left(f_1\theta_1 + \theta_1\,(1-\theta_1)\,\frac{n^2 s_1^2}{\delta_1^2}\right)^2 - f_1^2\right]. \tag{82}$$

Thus, in the presence of two strong discontinuities, the ratios of the densities on the front of "forward" ($\theta(s_0) = \theta_0$) and second "internal" ($\theta(s_1) = \theta_1$) discontinuities, where $0 < s_0 < s_1 < \infty$, are determined respectively from Eqs. (55) and (82). The parameters s_0 and s_1 are eigenvalues of sorts of the problem in question. They may be determined, e.g., by the numerical method using the "targeting" of boundary conditions (45) specified on the piston $s = 0$.

Let us consider the existence of two strong discontinuities (67) in greater detail. On the "forward"-discontinuity front, from (56) and (53) we obtain

$$C_{T_2}^2 = (\gamma-1)\,\frac{\hat{K}_0}{\hat{\tau}_0}\,f_2 = \theta_0\,(1-\theta_0)\,(\gamma-1)\,n^2 s_0^2\,\frac{\hat{K}_0}{\hat{\tau}_0}\,,\quad C_{i_2}^2 = \delta_2^2 f_2 = \frac{1-\theta_0}{\theta_0}\,n^2 s_0^2\,. \tag{83}$$

Expression (64) with account for (83) may be written as

$$\lambda_{\pm}^2(s_0) = \frac{1}{2}\,n^2 s_0^2\left[(\gamma-1)\,\frac{\hat{K}_0}{\hat{\tau}_0}\,\theta_0\,(1-\theta_0) + \right.$$

$$\left. + \frac{1-\theta_0}{\theta_0}\,\gamma \pm \sqrt{\left((\gamma-1)\,\frac{\hat{K}_0}{\hat{\tau}_0}\,\theta_0\,(1-\theta_0) - \frac{1-\theta_0}{\theta_0}\,\gamma\right)^2 + 4\,(\gamma-1)^2\,\frac{\hat{K}_0}{\hat{\tau}_0}\,(1-\theta_0)^2}\,\right]. \tag{84}$$

We transform the radicand:

$$(\gamma-1)^2\,\frac{\hat{K}_0}{\hat{\tau}_0}\,\theta_0^2\,(1-\theta_0)^2 - 2\gamma\,(\gamma-1)\,\frac{\hat{K}_0}{\hat{\tau}_0}\,(1-\theta_0)^2 + \left(\frac{1-\theta_0}{\theta_0}\,\gamma\right)^2 + 4\gamma\,(\gamma-1)\,\frac{\hat{K}_0}{\hat{\tau}_0}\,(1-\theta_0)^2 -$$

$$- 4\,(\gamma-1)\,\frac{\hat{K}_0}{\hat{\tau}_0}\,(1-\theta_0)^2 = \left((\gamma-1)\,\frac{\hat{K}_0}{\hat{\tau}_0}\,\theta_0\,(1-\theta_0) + \frac{1-\theta_0}{\theta_0}\,\gamma\right)^2 - 4\,(\gamma-1)\,\frac{\hat{K}_0}{\hat{\tau}_0}\,(1-\theta_0)^2\,.$$

From Eq. (55), multiplying by $4(1-\theta_0)$, we obtain

$$4(\gamma-1)\frac{\hat{K}_0}{\hat{\tau}_0}(1-\theta_0)^2 = \frac{4}{\theta_0^2}[(\gamma+1)\theta_0-(\gamma-1)](1-\theta_0) = 4(\gamma-1)\left(\frac{\gamma+1}{\gamma-1}-\frac{1}{\theta_0}\right)\left(\frac{1}{\theta_0}-1\right).$$

Setting $\delta_2 = 1/\theta_0$ and taking account of the last expressions, we represent (84) in the form

$$\lambda_\pm^2(s_0) = \frac{1}{2}n^2 s_0^2\left[1+\delta_2 \pm \sqrt{(1+\delta_2)^2 - 4(\gamma-1)(\delta_2-1)\left(\frac{\gamma+1}{\gamma-1}-\delta_2\right)}\right]. \quad (85)$$

We show that inequalities (67) always hold for the problem in question.

1. Let $\lambda_+^2(s_0)$ be more than $n^2 s_0^2$. Then (85) yields

$$1 < \frac{1}{2}(1+\delta_2) + \frac{1}{2}\sqrt{(1+\delta_2)^2 - 4(\gamma-1)(\delta_2-1)\left(\frac{\gamma+1}{\gamma-1}-\delta_2\right)}$$

and consequently

$$1-\delta_2 < \sqrt{(1+\delta_2)^2 - 4(\gamma-1)(\delta_2-1)\left(\frac{\gamma+1}{\gamma-1}-\delta_2\right)}\ .$$

Since we have $\delta_2 > 1$ on the discontinuity front, we arrive at the trivial inequality. The condition $\lambda_+^2(s_0) > n^2 s_0^2$ is observed.

2. Let $\lambda_-^2(s_0)$ be less than $n^2 s_0^2$. Then we obtain

$$1 > \frac{1}{2}(1+\delta_2) - \frac{1}{2}\sqrt{(1+\delta_2)^2 - 4(\gamma-1)(\delta_2-1)\left(\frac{\gamma+1}{\gamma-1}-\delta_2\right)}$$

and consequently

$$\delta_2 - 1 < \sqrt{(1+\delta_2)^2 - 4(\gamma-1)(\delta_2-1)\left(\frac{\gamma+1}{\gamma-1}-\delta_2\right)}\ . \quad (86)$$

From (86) we have $(1+\delta_2)^2 - 4(\gamma-1)(\delta_2-1)\left(\frac{\gamma+1}{\gamma-1}-\delta_2\right) > (\delta_2-1)^2$ or, after the transformations,

$$\delta_2 > (\gamma-1)(\delta_2-1)\left(\frac{\gamma+1}{\gamma-1}-\delta_2\right). \quad (87)$$

We compare the curves $y_1 = \delta_2$ and $y_2 = (\gamma-1)(\delta_2-1)\left(\frac{\gamma+1}{\gamma-1}-\delta_2\right)$. The function y_1 is a straight line and y_2 is a parabola; here, $y_2 = 0$ for $\delta_2 = 1$ and $\delta_2 = (\gamma+1)/(\gamma-1)$. The derivative has the form

$$\frac{dy_2}{d\delta_2} = 2[\gamma-(\gamma-1)\delta_2]\ .$$

It is clear that $dy_2/d\delta_1 = 0$ for $\delta_2 = \gamma/(\gamma-1)$. The maximum value is determined by the expression

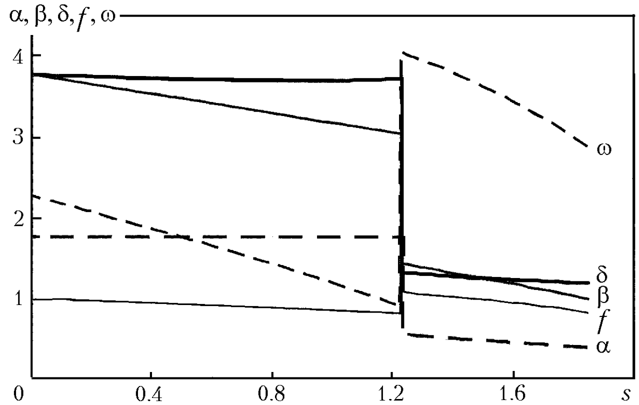

Fig. 1. Distribution of the functions α, β, δ, *f*, and ω in *s*, which characterizes the presence of two strong discontinuities successively propagating one after another from the piston ($s = 0$) to the value $s = s_0$. The position of the forward and internal discontinuities is characterized by the coordinates $s_0 = 1.844$ and $s_1 = 1.229$.

$$y_2\left(\frac{\gamma}{\gamma-1}\right)=\frac{1}{\gamma-1}\,.$$

The inequality

$$y_2\left(\frac{\gamma}{\gamma-1}\right)=\frac{1}{\gamma-1}<y_1\left(\frac{\gamma}{\gamma-1}\right)=\frac{\gamma}{\gamma-1}$$

yields that the y_1 line lies higher than the y_2 curve for any values of $\gamma > 1$, $1 \le \delta_2 \le (\gamma+1)/(\gamma-1)$. Consequently, inequality (87) holds, i.e., $\lambda_-^2(s_0) < n^2 s_0^2$.

It has been noted above that the numerical solution of the system of ordinary differential equations (57)–(62) in the region $0 \le s \le s_0$ with boundary conditions (50)–(55) may be carried out from the value $s = s_0$ toward $s = 0$ with "targeting" of piston condition (45). When inequalities (67) hold, the solution sought has the second discontinuity determined by relations (76)–(82) at a certain point $s = s_1$, $0 < s_1 < s_0$. To "target" one boundary condition (e.g., $f(0) = 1$) analogously to [11] we may use renormalization of the profiles of the quantities sought using similarity conversion. Indeed, we assume that we have selected a certain value of the parameter $s_0 = \overline{s}_0$ and the value $s_1 = \overline{s}_1$ in the case of (67). As a result of numerical integration for $s = 0$, we obtain certain values of the quantities sought, including the dimensionless functions of temperature and velocity $\underline{f}(0) = \overline{f}_0$ and $\alpha(0) = \overline{\alpha}_0$.

In the general case we have $\overline{f}_0 \ne 1$ and $\alpha(0) \ne \overline{\alpha}_0$. We select a "new" value — $T_{0*} = \overline{f}_0 T_0$ — instead of T_0 as the measurement scale. Then, using (38), we may write

$$s=\frac{m\overline{f_0^{1/2}}}{\rho_0\,(RT_{0*})^{1/2}\,t^{n}}\,,\quad f(s)=\frac{T\overline{f}_0}{T_{0*}t^{n_0}}\,,\quad \alpha(s)=\frac{v\overline{f_0^{1/2}}}{(RT_{0*})^{1/2}\,t^{\frac{1}{2}n_0}}\,,\quad \delta(s)=\frac{\rho}{\rho_0}\,,\quad \omega(s)=\frac{W\overline{f_0^{3/2}}}{\rho_0\,(RT_{0*})^{3/2}\,t^{\frac{3}{2}n_0}}\,. \tag{88}$$

From (88), we obtain

$$\overline{s}=\frac{m}{\rho_0\,(RT_{0*})^{1/2}\,t^{n}}\,,\quad \overline{f}(\overline{s})=\frac{T}{T_{0*}t^{n_0}}\,,\quad \overline{\alpha}(\overline{s})=\frac{v}{(RT_{0*})^{1/2}\,t^{\frac{1}{2}n_0}}\,,\quad \overline{\delta}(\overline{s})=\frac{\rho}{\rho_0}\,,\quad \overline{\omega}(\overline{s})=\frac{W}{\rho_0\,(RT_{0*})^{3/2}\,t^{\frac{3}{2}n_0}}\,, \tag{89}$$

where

$$\bar{s} = \frac{s}{f_0^{1/2}} ; \quad \bar{f}(\bar{s}) = \frac{f(s)}{f_0} ; \quad \bar{\alpha}(\bar{s}) = \frac{\alpha(s)}{f_0^{1/2}} ; \quad \bar{\delta}(\bar{s}) = \delta(s) ; \quad \bar{\omega}(\bar{s}) = \frac{\omega(s)}{f_0^{3/2}} ; \quad \bar{\beta}(\bar{s}) = \bar{\delta}(\bar{s}) \bar{f}(\bar{s}) . \tag{90}$$

After the numerical solution of system (57)–(62) in the region $0 \le s \le s_0$, we recalculate "new" functions from formulas (89). At the point $\bar{s} = 0$, we obtain

$$\bar{f}(0) = 1 , \quad \bar{\alpha}(0) = \frac{\overline{\alpha_0}}{f_0^{1/2}} .$$

The corresponding numerical example with two strong discontinuities of the quantities sought, which successively propagate one after another from the piston $s = 0$ to the value $s = s_0$, is presented in Fig. 1. We have considered the following initial parameters:

$$\gamma = \frac{5}{3} , \quad a_0 = \frac{5}{2} , \quad a_1 = \frac{3}{2} , \quad \frac{\hat{K}_0}{\hat{\tau}_0} = \frac{504}{25} , \quad n_0 = \frac{2}{3} , \quad n_1 = \frac{1}{3} , \quad n = \frac{4}{3} .$$

The values of $f(0) = 1$ and $\alpha(0) = \alpha_0 = 1.778285809$ have been prescribed for $s = 0$.

Thus, the above analysis shows that we always have two strong discontinuities propagating ahead of the piston.

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NOTATION

a_0 and a_1, exponents in the dependences of the thermal conductivity and the relaxation time on temperature; b_0 and b_1, exponents in the dependences of the thermal conductivity and the relaxation time on density; C_γ, velocity of sound; C_T, mass velocity of propagation of thermal disturbances at $\tau \neq 0$; D, velocity of the front of strong discontinuity of the quantities sought; $f = f(s)$, dimensionless temperature; $\hat{K}_0$, dimensionless combination of contents; $K = K(\rho, T)$, mass thermal conductivity; m, mass Lagrangian coordinate; p, pressure; R, universal gas constant; s, dimensionless "self-similar" variable; t, time; T, temperature; v, velocity; W, heat flux; $\alpha = \alpha(s)$, dimensionless velocity; $\beta = \beta(s)$, dimensionless pressure; $\gamma > 1$, constant ratio of specific heats; $\delta = \delta(s)$, dimensionless density; ε, specific internal energy; $\theta = \rho_1/\rho$, specific volume; $\lambda_\pm$, combinations of the "velocities of sound" C_T and C_γ; ρ_0, initial density; ρ, density; $\hat{\tau}_0$, dimensionless combination of constants; $\tau = \tau(\rho, T)$, time of relaxation of the heat-flux; $\omega = \omega(s)$, dimensionless heat flux. Subscripts: 1 and 2, quantities ahead of the discontinuity front and behind it.

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